The Poisson bracket on free null initial data for gravity

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Free initial data for general relativity on a pair of intersecting null hypersurfaces are well known, but the lack of a Poisson bracket and concerns about caustics have stymied the development of a constraint free canonical theory. Here it is pointed out how caustics and generator crossings can be neatly avoided and a Poisson bracket on free data is given. On sufficiently regular functions of the solution spacetime geometry this bracket matches the Poisson bracket defined on such functions by the Hilbert action via Peierls' prescription. The symplectic 2-form is also given in terms of free data.

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A constraint free canonical formulation of general relativity (GR) is of interest not least because at present the handling of constraints absorbs most of the effort invested in canonical approaches to quantizing gravity. Already in the 1960s free initial data for GR were identified on certain types of piecewise null hypersurfaces [1, 2, 3, 4], in particular on a "double null sheet". This is a compact hypersurface $\mathcal{N}$ consisting of two null branches, $\mathcal{N}_L$ and $\mathcal{N}_R$, swept out by the two future directed normal congruences of null geodesics (called *generators*) emerging from a spacelike 2-disk S_0, the branches being truncated on disks S_L and S_R before the generators form caustics (see Fig. 1).

Nevertheless a constraint free canonical theory was not constructed, for two reasons: First, the Poisson brackets of the free initial data were unknown. Second, in order that $\mathcal{N}$ not enter its own future, implying intractable constraints on the otherwise free initial data, the generators must not cross at interior points of $\mathcal{N}$ [5]. But excluding such crossings itself seemed to require intractable conditions on the data. Here a Poisson bracket on free data corresponding to the Hilbert action is presented, and a simple way to avoid caustics and generator crossings is pointed out.

The resulting framework seems ideal for attempting a semi-

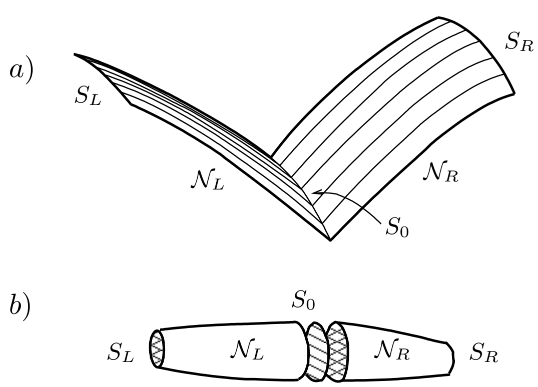

FIG. 1: a) A double null sheet in 2+1 dimensional spacetime. b) In 3+1 dimensional spacetime $\mathcal{N}$ is a 3-manifold consisting of two solid cylinders joined on a disk, here shown without regard to their embedding in spacetime.

classical proof of the Bousso entropy bound [6, 7, 8, 9] in the vacuum gravity case, since a branch $\mathcal{N}_A$ ($A = L$ or R) of $\mathcal{N}$ is a "light sheet" in the terminology of Bousso [9] provided the generators are not diverging at S_0.

Canonical GR using *constrained data* on double null sheets has been developed by several authors [10, 11, 12, 13]. Presumably the present Poisson brackets can be interpreted as Dirac brackets in those frameworks. Results on the brackets of part of the free data are given in Refs. [12, 14]. Reference [14] gives perturbation series in Newton's constant for the brackets of free data living on the bulk of $\mathcal{N}$ consistent with the present work, but no brackets of the surface data on S_0. Reference [12] presents distinct free data on the bulk of $\mathcal{N}$, which are claimed to form a canonically conjugate pair on the basis of a machine calculation of Dirac brackets. It would be interesting to see if they are conjugate according to the bracket obtained here.

A special chart $(v_A, \theta^1, \theta^2)$ will be used on each branch $\mathcal{N}_A$ of $\mathcal{N}$, with v_A a parameter along the generators and θ^a ($a = 1, 2$) constant along these. Since ∂_{v_A} is tangent to the generators it is null and normal to $\mathcal{N}_A$. The line element on $\mathcal{N}_A$ thus takes the form

$$ds^2 = h_{ab} d\theta^a d\theta^b, \tag{1}$$

with no dv terms. v_A is taken proportional to the square root of $\rho \equiv \sqrt{\det h}$, the area density in θ coordinates on 2D cross sections of $\mathcal{N}_A$, and normalized to 1 at S_0. Thus $\rho = \rho_0(\theta^1, \theta^2) v_A^2$, with ρ_0 the area density on S_0. Any affine parameter η on the generators is related to v by [15]

$$0 = R[\partial_v, \partial_v] = \frac{2}{v}\frac{d}{dv} \ln\left|\frac{d\eta}{dv}\right| + \frac{1}{4}\partial_v e_{ab} \partial_v e^{ab}, \tag{2}$$

a vacuum Einstein equation equivalent to the "focusing equation" (9.2.32 in [5]). Here $e_{ab} = h_{ab}/\rho$, a unit determinant, symmetric 2×2 matrix.

At caustic points $v^2 \equiv \rho/\rho_0$ vanishes, so the caustic *free* $\mathcal{N}$ are represented by initial data on coordinate domains in which $v > 0$. In the absence of caustics generators can still cross on $\mathcal{N}$ but the crossing points can be "unidentified": A spacetime which is locally isometric to a neighborhood of $\mathcal{N}$, but in which the generators do not cross, may be constructed by pulling the metric of the original spacetime back to the

normal bundle of S_0 using the exponential map (Ref. [15], Appendix B). The exclusion of caustics and crossings thus requiers no restriction on the data at $v > 0$. In particular it does not restrict the scope of the present work to weak fields.

Since $\mathcal{N}_A$ is caustic free $d\theta^1 \wedge d\theta^2$ is degenerate only along generators. $(v_A, \theta^1, \theta^2)$ is thus a good chart provided $dv_A \neq 0$ on the generators. For smooth S_0, in a smooth vacuum solution, this is so if the generators are converging everywhere on S_0 (v decreasing away from S_0), since by the focusing equation (2) v continues to decrease until a caustic is reached, and also if the generators are diverging everywhere on S_0 but are truncated before they begin to reconverge. In these cases e_{ab} induced by the spacetime geometry is smooth in v and θ^a. Conversely, if (A) v is not strictly constant on a generator, and (B) e_{ab} is smooth in v, then by (2) $dv \neq 0$ along the generator. The initial data will satisfy conditions A and B on all generators, ensuring that (v, θ^1, θ^2) is a good chart.

Sachs [1] showed (modulo convergence issues) that e_{ab}, specified on $\mathcal{N}$ as a function of an affine parameter on the generators, together with additional data on S_0, is free initial data determining the geometry of a spacetime region to the future of $\mathcal{N}$. Here we assume that any Sachs data without caustics determines a unique maximal Cauchy development $D[\mathcal{N}]$ of all of $\mathcal{N}$, and that if the data depend smoothly on a parameter the solution does as well. Existence, uniqueness, and smooth dependence on parameters have been proved rigorously in a neighborhood of S_0 [16].

We will use similar data, including e_{ab}, given on $\mathcal{N}$ as a smooth function of v; and ρ_0, $\lambda = -\ln|n_L \cdot n_R|$, and

$$\tau_a = \frac{n_L \cdot \nabla_a n_R - n_R \cdot \nabla_a n_L}{n_L \cdot n_R} \tag{3}$$

specified on S_0. [Here $n_A = \partial_{v_A}$ is the tangent to the generators of $\mathcal{N}_A$, and inner products $(\cdot)$ are evaluated using the spacetime metric.] v_A ranges from 1 on S_0 to $\bar{v}_A(\theta)$ on S_A. $\bar{v}_A(\theta)$, which is another datum, is required to be > 0 and $\neq 1$. The data are smooth functions of the θ^a, which range over the unit disk $(\theta^1)^2 + (\theta^2)^2 \leq 1$. Any valuation of these data determine, via (2), Sachs data free of caustics and thus, according to our asumptions, a solution to GR unique up to diffeomorphisms [15].

However, because $\mathcal{N}$ has a boundary, not all infinitesimal diffeomorphisms are degeneracy vectors of the symplectic 2-form on $\mathcal{N}$ [15]. Two further data on S_0, s_L^m and s_R^i, measure diffeomorphisms which are non-gauge in this sense. $y_A^k = s_A^k(\theta)$ is the position of the endpoint on $S_A \subset \partial\mathcal{N}$ of the generator (θ^1, θ^2), in a fixed chart y_A on S_A. Since s_A may be varied independently of the other data by diffeomorphisms the complete data set is still free. We shall return to the question of the significance of the s_A.

In sum, the data consist of 10 real C^∞ functions, ρ_0, λ, τ_a, $\bar{v}_A$, and s_A^k, on the unit 2-disk with $\bar{v}_A > 0$ and $\neq 1$, and two C^∞, real, symmetric, unimodular 2×2 matrix valued functions (e_{ab} on $\mathcal{N}_L$ and $\mathcal{N}_R$) on the domains $\{(\theta^1)^2 + (\theta^2)^2 \leq 1, \min(1, \bar{v}_A(\theta)) \leq v_A \leq \max(1, \bar{v}_A(\theta))\}$, $A = L, R$ which match at $v_L = v_R = 1$ (i.e. on S_0). Our phase space is the space of valuations of these data.

An alternative representation of e_{ab} can be obtained by expressing the degenerate line element (1) on $\mathcal{N}$ in terms of the complex coordinate $z = \theta^1 + i\theta^2$:

$$\begin{aligned} ds^2 &= h_{ab} d\theta^a d\theta^b \\ &= \rho(1 - \mu\bar{\mu})^{-1}[dz + \mu d\bar{z}][d\bar{z} + \bar{\mu} dz], \end{aligned} \tag{4}$$

with μ a complex number valued field of modulus less than 1 (sometimes called the *Beltrami differential*). μ encodes the two real degrees of freedom of $e_{ab} = h_{ab}/\rho$. This parametrization of e_{ab} also works when e_{ab} is not real, but then μ and $\bar{\mu}$ are no longer complex conjugates.

Finding the Poisson bracket on initial data by inverting the (gauge fixed) symplectic 2-form, or other conventional approaches, turns out to be difficult. But what is ultimately required of the bracket is that it gives the correct Poisson brackets for observables. We shall content ourselves with finding a bracket that satisfies this criterion. Observables will be defined as diffeomorphism invariant functionals of the metric $F[g]$ with C^∞ functional derivatives $\delta F/\delta g_{\mu\nu}$ of compact support contained in the interior of the Cauchy development $D[\mathcal{N}]$. The Poisson brackets of these observables may be defined via Peierls' [17] covariant formula in terms of the action and Green's functions [15].

To match the Peierls bracket on observables $\{\cdot,\cdot\}_\bullet$ need only be *almost* inverse to the symplectic 2-form $\omega_{\mathcal{N}}$ on $\mathcal{N}$. Specifically, let g be a metric satisfying the field equations and let L_g^0 be the space of perturbations of the metric that satisfy the field equations linearized about g and vanish in a neighborhood of $\partial\mathcal{N}$. Then $\{\cdot,\cdot\}_\bullet$ matches the Peierls bracket on observables (at g) if

$$\delta\varphi = \omega_{\mathcal{N}}[\{\varphi,\cdot\}_\bullet, \delta] \quad \forall\, \delta \in L_g^0. \tag{5}$$

for all integrals, φ, of initial data against smooth test functions on $\mathcal{N}$ that vanish in a neighborhood of $\partial\mathcal{N}$ [15].

We will use the symplectic 2-form of the Hilbert action, and we will require $\{\cdot,\cdot\}_\bullet$ to be causal: data at $p \in \mathcal{N}$ must commute with data outside the causal domain of influence of p, domain which on $\mathcal{N}$ reduces to just the generator(s) through p [15]. Then $\omega_{\mathcal{N}}[\{\varphi,\cdot\}_\bullet, \delta]$ in (5) can be expressed in terms of the initial data as $\Omega_{\mathcal{N}}[\{\varphi,\cdot\}_\bullet, \delta]$ where $\Omega_{\mathcal{N}} = \Omega_L + \Omega_R$ with [15]

$$\begin{aligned} 16\pi G \Omega_A[\delta_1, \delta_2] = \int_{S_0} d^2\theta \Big\{ & \delta_1 \lambda \delta_2 \rho_0 + \delta_1 \tilde{\tau}_{A\,k} \delta_2 s_A^k \\ & + \frac{1}{4} \delta_1^y \rho_0 \partial_v e_{ab} \delta_2^y e^{ab} \\ & + \frac{1}{2} \rho_0 \int_1^{\bar{v}} v^2 \delta_1^\circ e^{ab} \partial_v \delta_2^\circ e_{ab} dv \Big\} \\ & - (1 \leftrightarrow 2). \end{aligned} \tag{6}$$

Here $\tilde{\tau}_{R\,i} dy_R^i = \rho_0[d\lambda - \tau]$ and $\tilde{\tau}_{L\,m} dy_L^m = \rho_0[d\lambda + \tau]$; $\delta^y = \delta - \pounds_{\xi_\perp}$, where $\xi_\perp = \delta s_A^k \partial_{y_A^k}$ and the partial derivative

$\partial_{y_A^k}$ is taken at constant v; and $\delta^\circ = \delta^y - \frac{1}{2}\delta^y \ln \rho_0\, v\partial_v$. In calculating (6) no boundary terms were added to the Hilbert action because the Peierls bracket, which determines the brackets of observables, is unaffected by such terms. Equation (6) is consistent with [18].

Equation (5) does not determine the bracket uniquely, nor does it guarantee that it satisfies the Jacobi relations. Here a unique Poisson bracket is obtained by defining a set C of variations of the data containing those corresponding to spacetime metric variations in L_g^0 and imposing

$$\delta\varphi = \Omega_{\mathcal{N}}[\{\varphi,\cdot\}_\bullet,\delta] \quad \forall\, \delta \in C \quad \text{and} \quad \{\varphi,\cdot\}_\bullet \in C, \tag{7}$$

for all φ obtained by smearing data with *any* C^∞ test function on $\mathcal{N}$ or S_0. The first condition ensures agreement with the Peierls bracket; the second that the Jacobi relations hold.

C consists of all *complex* variations δ of the data such that (A) $\delta\bar{\mu}$ is smooth on $\mathcal{N}$ while $\delta\mu$ is smooth on $\mathcal{N}_L - S_0$, $\mathcal{N}_R - S_0$, and S_0, with possible jump dicontinuities between them, and (B) δ leaves invariant on S_A both ρ_A, the area density in the y_A chart, and μ_A, the Beltrami differential in the complex chart $y_A^1 + iy_A^2$. ($\delta\bar{\mu}_A$ need *not* vanish on S_A.)

The use of the space C of complex perturbations on the real phase space is strange but seems difficult to avoid. Note however that all hamiltonian vectors defined by the $\bullet$ bracket satisfying (7) preserve the reality of observables, and of the metric on the interior of $D[\mathcal{N}]$.

Because of the identity

$$\tilde{\tau}_{R\,i}\partial_a s_R^i + \tilde{\tau}_{L\,m}\partial_a s_L^m = 2\rho_0\partial_a\lambda \tag{8}$$

$\Omega_{\mathcal{N}}$ is degenerate with respect to variations of the data due to diffeomorphisms of the θ chart on S_0. This degeneracy can be removed by extending the phase space by making $\tilde{\tau}_R$ and $\tilde{\tau}_L$ independent, and treating (8) as a constraint (which generates diffeomorphisms of θ [15]). Then (7) defines the $\bullet$ brackets of the data uniquely as two point distributions on S_0 and $\mathcal{N}$.

[A description without *any* constraint may be obtained by eliminating $\tilde{\tau}_L$ via (8), and s_L via the gauge fixing $\theta = y_L$. The Dirac brackets of the remaining data are then identical to their extended phase space $\bullet$ brackets [15].]

Solving (7) by a procedure like that of [15] yields:

$$\{\rho_0(\boldsymbol{\theta}_1),\lambda(\boldsymbol{\theta}_2)\}_\bullet = 8\pi G\delta^2(\boldsymbol{\theta}_2-\boldsymbol{\theta}_1), \tag{9}$$

$$\{s_A^i(\boldsymbol{\theta}_1),\tilde{\tau}_{B\,j}(\boldsymbol{\theta}_2)\}_\bullet = 16\pi G\delta_{AB}\delta_j^i\delta^2(\boldsymbol{\theta}_2-\boldsymbol{\theta}_1), \tag{10}$$

and s_A and ρ_0 commute with all other data.

The brackets between $\tilde{\tau}_R$, $\tilde{\tau}_L$, and λ are

$$\{\lambda(\boldsymbol{\theta}_1),\lambda(\boldsymbol{\theta}_2)\}_\bullet = 0, \tag{11}$$

$$\{\lambda(\boldsymbol{\theta}),\tau_R[f]\}_\bullet \;=\; 8\pi G\Big[\frac{\pounds_f\mu}{(1-\mu\bar{\mu})^2}(\partial_{v_R}\bar{\mu}-\partial_{v_L}\bar{\mu})\Big]_{\boldsymbol{\theta}}, \tag{12}$$

$$\begin{aligned}\{\tau_R[f_1],\tau_R[f_2]\}_\bullet \;=\; & -16\pi G\int_{S_0}\frac{1}{(1-\mu\bar{\mu})^2}\pounds_{f_1}\mu(\epsilon\pounds_{f_2}\bar{\mu} \\ & -\pounds_{f_2}\epsilon\partial_{v_R}\bar{\mu}) - (1\leftrightarrow 2),\end{aligned} \tag{13}$$

$$\begin{aligned}\{\tau_R[f],\tau_L[g]\}_\bullet \;=\; & 16\pi G\int_{S_0}\frac{1}{(1-\mu\bar{\mu})^2}\pounds_f\mu(\epsilon\pounds_g\bar{\mu} \\ & -\pounds_g\epsilon\partial_{v_L}\bar{\mu}) - (f,R\leftrightarrow g,L),\end{aligned} \tag{14}$$

the rest being obtainable from these by exchanging L and R. ϵ is the area form $\rho_0 d\theta^1\wedge d\theta^2$, $\tau_R[f] = \int_{S_0}\tilde{\tau}_{R\,i}f^i d^2\theta$ with $f^i(\theta)$ test functions independent of the data, and $\tau_L[g]$ is defined similarly in terms of test functions g^m. f^i and g^m define vector fields $f = f^i\partial_{y_R^i}$ and $g = g^m\partial_{y_L^m}$, and thus Lie derivatives. The only unusual one is

$$\pounds_f\mu = f^z\partial_z\mu + f^{\bar{z}}\partial_{\bar{z}}\mu - \mu[\partial_z f^z - \partial_{\bar{z}}f^{\bar{z}}] - \mu^2\partial_z f^{\bar{z}} + \partial_{\bar{z}}f^z$$

The brackets between μ and $\bar{\mu}$ are as follows: If $\mathbf{1},\mathbf{2}$ denote the (v,θ) coordinates of any pair of points on $\mathcal{N}$

$$0 = \{\mu(\mathbf{1}),\mu(\mathbf{2})\}_\bullet = \{\bar{\mu}(\mathbf{1}),\bar{\mu}(\mathbf{2})\}_\bullet. \tag{15}$$

When $\mathbf{1}$ and $\mathbf{2}$ do not lie on the same branch also $\{\mu(\mathbf{1}),\bar{\mu}(\mathbf{2})\}_\bullet = 0$, but if $\mathbf{1}$ and $\mathbf{2}$ do lie on the same branch, $\mathcal{N}_A$, then

$$\begin{aligned}\{\mu(\mathbf{1}),\bar{\mu}(\mathbf{2})\}_\bullet \;=\; & 4\pi G\,\frac{1}{\rho_0}\delta^2(\boldsymbol{\theta}_2-\boldsymbol{\theta}_1)\,H(\mathbf{1},\mathbf{2})\Big[\frac{1-\mu\bar{\mu}}{v_A}\Big]_{\mathbf{1}} \\ & \times\Big[\frac{1-\mu\bar{\mu}}{v_A}\Big]_{\mathbf{2}} e^{\int_{\mathbf{1}}^{\mathbf{2}}(\bar{\mu}d\mu-\mu d\bar{\mu})/(1-\mu\bar{\mu})},\end{aligned} \tag{16}$$

where if $\mathbf{1}$ and $\mathbf{2}$ lie on the same generator the integral runs along the generator segment from $\mathbf{1}$ to $\mathbf{2}$, and H is a step function equal to 1 if $\mathbf{1}$ lies on or between S_0 and $\mathbf{2}$, and 0 otherwise. Here and elsewhere the coefficient of $\delta^2(\boldsymbol{\theta}_2-\boldsymbol{\theta}_1)$ is extended continuously to $\boldsymbol{\theta}_2\neq\boldsymbol{\theta}_1$.

There remain the brackets of μ and $\bar{\mu}$ with the S_0 data λ, $\tilde{\tau}_R$, and $\tilde{\tau}_L$. For $\mathbf{1}$ on $\mathcal{N}_R - S_0$ (i.e. $v_{R\,1}\neq 1$)

$$\{\mu(\mathbf{1}),\lambda(\boldsymbol{\theta}_2)\}_\bullet = 4\pi G\frac{1}{\rho_0}\delta^2(\boldsymbol{\theta}_2-\boldsymbol{\theta}_1)[v_R\partial_{v_R}\mu]_{\mathbf{1}}, \tag{17}$$

$$\{\mu(\mathbf{1}),\tau_R[f]\}_\bullet = 8\pi G\Big[2\pounds_f\mu - \frac{\pounds_f\rho_0}{\rho_0}v_R\partial_{v_R}\mu\Big]_{\mathbf{1}}, \tag{18}$$

$$\{\mu(\mathbf{1}),\tau_L[g]\}_\bullet = 0, \tag{19}$$

while for $\mathbf{1}$ on S_0

$$\{\mu(\mathbf{1}),\lambda(\mathbf{2})\}_\bullet = 0, \tag{20}$$

$$\{\mu(\mathbf{1}),\tau_R[f]\}_\bullet = 8\pi G[\pounds_f\mu]_{\mathbf{1}}, \tag{21}$$

$$\{\mu(\mathbf{1}),\tau_L[g]\}_\bullet = 8\pi G[\pounds_g\mu]_{\mathbf{1}}. \tag{22}$$

On the other hand, for all $\mathbf{1} \in \mathcal{N}_R$ (including $\mathbf{1} \in S_0$)

$$\begin{aligned}\{\bar{\mu}(\mathbf{1}), \lambda(\boldsymbol{\theta}_2)\}_\bullet \;=\; & 4\pi G \frac{1}{\rho_0} \delta^2(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)\Big[(v_R \partial_{v_R} \bar{\mu})_\mathbf{1} \\ & + \Big(\frac{1}{v_R}\Big)_\mathbf{1} e^{2\int_\mathbf{1}^\mathbf{2} (\mu d\bar{\mu})/(1-\mu\bar{\mu})} (\partial_{v_L} \bar{\mu})_\mathbf{2}\Big],\end{aligned} \tag{23}$$

$$\begin{aligned}\{\bar{\mu}(\mathbf{1}), \tau_R[f]\}_\bullet \;=\; & 8\pi G \Big[\Big(2\pounds_f \bar{\mu} - \frac{\pounds_f \rho_0}{\rho_0} v_R \partial_{v_R} \bar{\mu}\Big)_\mathbf{1} \\ & -(\pounds_f \bar{\mu})_{\mathbf{1}_0} \Big(\frac{1}{v_R}\Big)_\mathbf{1} e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}} (\mu d\bar{\mu})/(1-\mu\bar{\mu})}\Big],\end{aligned} \tag{24}$$

$$\begin{aligned}\{\bar{\mu}(\mathbf{1}), \tau_L[g]\}_\bullet \;=\; & 8\pi G \Big[\pounds_g \bar{\mu} - \frac{\pounds_g \rho_0}{\rho_0} \partial_{v_L} \bar{\mu}\Big]_{\mathbf{1}_0} \\ & \times \Big(\frac{1}{v_R}\Big)_\mathbf{1} e^{-2\int_{\mathbf{1}_0}^{\mathbf{1}} (\mu d\bar{\mu})/(1-\mu\bar{\mu})},\end{aligned} \tag{25}$$

where $\mathbf{1}_0 \in S_0$ is the base point of the generator through $\mathbf{1}$. Exchanging L and R in (17)–(25) gives the corresponding brackets for $\mathbf{1}$ on $\mathcal{N}_L$.

Finally, the brackets of $\bar{v}_A$ follow from the preceeding brackets and the fact that, by (7), $\rho_A = \rho_0 \bar{v}_A^2 |\det \partial_a s_A^k|^{-1}$ at given y_A commutes with everything. Alternatively, $\bar{v}_A(\theta)$ may be replaced as a phase coordinate by $\rho_A(y_A)$.

Direct calculations confirm that these expressions for the brackets satisfy the Jacobi relations, that they are invariant under diffeomorphisms of the (arbitrarily chosen) coordinates y_R^i, y_L^m and θ^a [15], and that the constraint (8) generates diffeomorphisms of the θ^a [15].

Strangely, the brackets do not preserve the reality of e_{ab}, i.e. the complex conjugacy of μ and $\bar{\mu}$. An analytic functional F of the data is real on real data iff it equals $\bar{F}$, its *formal complex conjugate*, obtained by exchanging μ and $\bar{\mu}$, leaving the S_0 data untouched, and replacing numerical coefficients by their complex conjugates. $\{F, \cdot\}_\bullet$ preserves the reality of the data for all formally real F iff the bracket itself is real in the sense that it equals the formal complex conjugate bracket $\{\varphi, \chi\}_{\bullet c.c.} \equiv \overline{\{\bar{\varphi}, \bar{\chi}\}_\bullet}$. But this is not so: $\overline{\{\mu(\mathbf{1}), \bar{\mu}(\mathbf{2})\}_\bullet} \neq \{\bar{\mu}(\mathbf{1}), \mu(\mathbf{2})\}_\bullet$. Nevertheless, on observables the bracket is real, as it reproduces the real Peierls bracket. In fact, one may resolve the $\bullet$ bracket into (formal) real and imaginary parts, $\{\cdot,\cdot\}_\bullet = \{\cdot,\cdot\}_R + i\{\cdot,\cdot\}_I$, and one finds that

$$\begin{aligned}\{\cdot,\cdot\}_I \;=\; & \frac{i}{8\pi G} \sum_{A=L,R} \int_{S_A} \epsilon \frac{1}{(1-\mu_A\bar{\mu}_A)^2} [\{\cdot, \bar{\mu}_A\}_\bullet \overline{\{\bar{\mu}_A, \bar{\cdot}\}_\bullet} \\ & - \overline{\{\bar{\cdot}, \bar{\mu}_A\}_\bullet} \{\bar{\mu}_A, \cdot\}_\bullet].\end{aligned} \tag{26}$$

In agreement with causality, $\{\bar{\mu}_A(q), \cdot\}_\bullet$ for $q \in S_A$ is a gravitational wave pulse that skims along $\mathcal{N}_A$ without entering the interior of $D[\mathcal{N}]$. It does not affect the metric there (Ref. [15], Appendix C), so neither does $\{\phi, \cdot\}_I$ for any datum ϕ.

$\{\cdot,\cdot\}_R$ is the pre-Poisson bracket $\{\cdot,\cdot\}_\circ$ of [15], which does not satisfy the Jacobi relations, so the imaginary part (26) is necessary. One may reverse its sign, but given the action there seems to be little, if any, further freedom in the bracket. Adding boundary terms to the action, which does not affect the Peierls bracket, might alter $\{\cdot,\cdot\}_\bullet$.

The data s_A, $A = L, R$ seem unphysical as they do not affect the geometry of the solution, yet they participate in the Poisson bracket. In fact s_A may be replaced almost entirely by μ_A, which commutes with everything. μ and μ_A together determine $s_A : \theta \mapsto y_A$ up to the three parameter group of conformal maps of the unit disk to itself. Moreover μ_A can always be set to zero by a suitable choice of y_A chart. The remaing three degrees of freedom in s_A are determined by the boundary values of s_A on ∂S_0, which commute with all data on the interior of $\mathcal{N}$. Of course, were S_0 a 2-sphere instead of a disk no boundary values would be available to fix the conformal automorphisms.

ρ_0, s_L, s_R and μ qualify as "configuration variables" since they form a maximal commuting set among functionals of the data. [We regard $\rho_L(y_L)$ and $\rho_R(y_R)$, which commute with everything, as fixed]. A quantization may thus be attempted in terms of wave functionals depending on ρ_0, s_L, s_R, and μ, but annihilated by $\delta/\delta\bar{\mu}$.

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